SIMULTANEOUS MEASUREMENT OF NON-COMMUTING OBSERVABLES

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It is shown that the full unknown state of a spin-$\frac{1}{2}$ system, S, which, within Born's statistical interpretation, is meant as the state of an ensemble of identically prepared systems and is described by its density matrix, can be determined with a simultaneous measurement with the help of an "assistant" system A whose initial state is known. The idea is to let S and A interact with each other in a known way during a proper interaction time τ, to measure simultaneously two observables, one of S and one of A and their correlation. One thus determines the three unknown components of the polarization vector of S by means of repeated experiments using a unique setting. In this way one can measure simultaneously all the non-commutative observables of S, which might seem prohibited in quantum mechanics.

Keywords: State determination; Quantum measurement; Two-level system; Coherent state.

1. Introduction

The determination of the unknown state of a quantum system is one of the most important issues in the field of quantum information.[1,2] This determination involves a measurement process in which a macroscopic system, apparatus, is coupled to the quantum system; during this process the state of both the apparatus and the system is modified.[3] For instance, as currently described in many textbooks, the z-component of the polarization vector of a spin-$\frac{1}{2}$system, S, can statistically be determined by means of a repeated Stern-Gerlach experiment. In this process, the x- and y-component of the polarization vector are destroyed as a consequence of the non-commutation of the spin operator in the transversal directions. Other experimental settings seem therefore necessary to measure the unknown polarization vector of S. Its three components are represented by incompatible observables, the Pauli operators, and their direct determination requires three macroscopic apparatuses, differing by a change of orientation of the magnets and detectors. Likewise, the state of any two-level system, represented by a 2×2 density matrix $\hat{\rho}$ can be fully determined only through measurement of three linearly independent observables which do not commute and cannot be simultaneously measured.

Nevertheless, we will prove that the whole unknown density matrix of such a system

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S, in particular the full polarization vector of a spin-$\frac{1}{2}$ system, can be determined indirectly by means of a set of measurements performed simultaneously on S and an auxiliary system, A, which we term the assistant. The strategy is the following: initially S is in the unknown state that we wish to determine, while the assistant A is in some known state. During some time interval S and A interact in a known fashion. Their joint state is modified, involves correlations and keeps memory of the initial state of S. A simultaneous measurement of one of the observables of S and A is then performed. Repeating this process provides then three statistical data: the expectation values of these observables and their correlation. We will show that one can infer the three components of the initial polarization vector of S from the three data.

There are two approaches to this problem. Either using another two-level assistant, for solid state applications[5] or using an electromagnetic field as an assistant, for quantum optics applications.

After defining the general problem, we illustrate, in section 3, the first approach, i.e. using another spin-$\frac{1}{2}$ system as an assistant in a known pure state to determine the initial state of S. Then, in section 4, we show that it is also possible to use an electromagnetic field in a coherent state to determine the whole elements of the unknown density matrix of S.

2. Statement of the Problem

The idea of mapping the state $\hat{\rho}$ of an unknown spin-$\frac{1}{2}$ system, S, onto a single observable of S+A system by using another system in a known state $\hat{R}$, A, was first proposed by D'Ariano.[4] It was explicitly implemented in a dynamic form in Ref. 5.

The state of the composite system $S + A$, which is tested, is

$$\hat{\mathcal{R}}_\tau = \hat{U}\hat{\mathcal{R}}_0\hat{U}^\dagger, \tag{1}$$

where the initial state of $S + A$ is $\hat{\mathcal{R}}_0 = \hat{R} \otimes \hat{\rho}$ and the evolution operator is $\hat{U} = e^{-i\hat{H}\tau}$. Therefore, the dynamics of the system yields the required mixing of $\hat{\rho}$ and $\hat{R}$ and the simplest possible non-degenerate observable of the composite system S+A, $\hat{\Omega}$, can be chosen as a factorized quantity

$$\hat{\Omega} = \hat{\omega} \otimes \hat{o}, \tag{2}$$

where the observable $\hat{\omega}$ and $\hat{o}$ pertain to S and A respectively. Then the spectral decomposition of $\hat{\omega}$ and $\hat{o}$ can be used to construct the projection operator $\hat{P}_\alpha$ of $\hat{\Omega}$

$$\hat{\omega} = \sum_{i=1}^{m} \omega_i \hat{\pi}_i, \qquad \hat{o} = \sum_{a=1}^{n} o_a \hat{p}_a, \tag{3}$$

where $\hat{p}_a$ and $\hat{\pi}_i$ are eigen projectors of the observables $\hat{o}$ and $\hat{\omega}$ respectively. Therefore, projection operator $\hat{P}_\alpha$ with $\alpha \equiv (i, a)$ takes the form

$$\hat{P}_\alpha \equiv \hat{P}_{ia} = \hat{\pi}_i \otimes \hat{p}_a. \tag{4}$$

Repeated measurements of $\hat{\Omega}$ which means repeated simultaneous measurements of $\hat{\omega}$ and $\hat{o}$, determines the joint probabilities to observe ω_i for S and o_a for A

$$P_\alpha \equiv P_{ia} = \text{Tr}[\hat{\mathcal{R}}_\tau(\hat{\pi}_i \otimes \hat{p}_a)], \tag{5}$$

where $\hat{\mathcal{R}}_\tau$ is defined in (Eq. 1). In fact the numbers P_α are the diagonal elements of $\hat{U}^\dagger(\hat{\rho} \otimes \hat{R})\hat{U}$ in the factorized basis which diagonalizes $\hat{\omega}$ and $\hat{o}$.

The whole elements of the density matrix $\hat{\rho}$ can be determined by the mapping $\hat{\rho} \to P_\alpha$. Like the idea of finding a universal observable,[4] if $\hat{H}$ couples S and A properly, this mapping will be expected to be invertible for $n \geq m$. We shall see that even simple interactions can achieve this condition. For given observables $\hat{\omega}$ of S and $\hat{o}$ of A and for a known initial state $\hat{R}$, the precision of this procedure relies on the ratio between the experimental uncertainty of P_α and the resulting uncertainty on $\hat{\rho}$, which can be characterized by the determinant, Δ, of the transformation (5). For $\Delta = 0$ it is impossible to determine $\hat{\rho}$ from P_α. This means, the system is unstable with respect to small errors made during experimental determination of P_α or equivalently $(\langle\hat{\sigma}_z\rangle, \langle\hat{s}_z\rangle, \langle\hat{s}_z\hat{\sigma}_z\rangle)$. Therefore, the Hamiltonian $\hat{H}$ and time interval τ should be chosen so as to maximize $|\Delta|$ over all possible unitary transformations.

3. Spin-$\frac{1}{2}$ Assistant in a Known Pure State

In this section we illustrate the above ideas by studying a two-level system S, namely, an spin-$\frac{1}{2}$ system in a known pure state. The density matrix of a spin-$\frac{1}{2}$ can be represented with the help of Pauli matrices. The determination of $\hat{\rho}$ corresponds to determination of the elements of the polarization vector, $\vec{\rho}$. we let S and A interact during the time interval τ. The observables $\hat{\omega}$ and $\hat{o}$ to be measured are the z-components of spin of S and A and are determined by $\hat{\sigma}_z$ and $\hat{s}_z$ respectively. The projection operators are

$$\hat{\pi}_i = \frac{1}{2}(\hat{1} + \hat{\sigma}_z), \quad \hat{p}_a = \frac{1}{2}(\hat{1} + \hat{s}_z), \tag{6}$$

for i and a equal to ± 1. Experiments will determine the four joint probabilities $P_\alpha = P_{++}, P_{+-}, P_{-+}, P_{--}$. These probabilities are related to the three real parameters $\vec{\rho}$ of $\hat{\rho}$ by inserting (Eq. 6) and $\hat{R} = \frac{1}{2}(\hat{1} + \hat{s}_z)$ into (Eq. 5).

$$P_\alpha = u_\alpha + \vec{v}_\alpha \cdot \vec{\rho}, \tag{7}$$

where

$$u_\alpha = \frac{1}{2}[\hat{U}(\hat{1} \otimes \hat{R})\hat{U}^\dagger]_{\alpha,\alpha}, \quad \vec{v}_\alpha = \frac{1}{2}[\hat{U}(\vec{\hat{\sigma}} \otimes \hat{R})\hat{U}^\dagger]_{\alpha,\alpha}, \tag{8}$$

with $\alpha = \{ia\} = \{++, +-, -+, --\}$ and matrix elements have been represented in the standard representation of the Pauli matrices, $\vec{\hat{\sigma}}$ and $\vec{\hat{s}}$. The probabilities P_α should be positive and normalized for any the density matrix, $\hat{\rho}$, such that $\vec{\rho}^2 \leq 1$. These conditions imply that

$$u_\alpha \geq |v_\alpha|, \qquad \sum_\alpha u_\alpha = 1, \qquad \sum_\alpha \vec{v}_\alpha = 0. \tag{9}$$

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The determinant of transformation $\hat{\rho} \to P_\alpha$ can be either

$$\vec{v}_{++} \cdot (\vec{v}_{+-} \times \vec{v}_{+-}), \tag{10}$$

or any other permutations of three of the vectors $\vec{v}_{++}$, $\vec{v}_{+-}$, $\vec{v}_{-+}$ and $\vec{v}_{--}$. Therefore, the determinant of the transformation is four times the volume of the parallelepiped made by three of these vectors. For example,

$$\Delta = 4\vec{v}_{++} \cdot (\vec{v}_{+-} \times \vec{v}_{+-}). \tag{11}$$

If the unitary evolution operator $\hat{U}$ is such that vectors $\vec{v}_\alpha$ are not coplanar, the transformation (Eq. 7) is invertible and one can determine $\vec{\rho}$ from the set of P_α. Alternatively, $\hat{\rho}$ is deduced from $\langle \hat{\sigma}_z \rangle$, $\langle \hat{s}_z \rangle$ and $\langle \hat{s}_z \hat{\sigma}_z \rangle$ at time τ. The notation $\hat{s}_z \hat{\sigma}_z$ is used for simplicity instead of $\hat{s}_z \otimes \hat{\sigma}_z$ which lives in the common Hilbert space of S and A. $\hat{\sigma}_z$, $\hat{s}_z$ and $\hat{s}_z \hat{\sigma}_z$ can be simultaneously measured and are in one to one correspondence with the set of probabilities P_α.

We first look for the upper bound of the determinant of transformation (Eq. 7), $|\Delta|$ implied by the conditions (Eq. 9). First we note that $|\Delta|$ increases with $|\vec{v}_\alpha|$ for each α. We therefore maximize Δ^2 under the constraints

$$\sum_\alpha |\vec{v}_\alpha| = 1, \quad \sum_\alpha \vec{v}_\alpha = 0. \tag{12}$$

This yields a symmetric solution for all these vectors

$$u_\alpha = |\vec{v}_\alpha| = \frac{1}{4}, \quad \cos(\vec{v}_\alpha, \vec{v}_\beta) = \frac{\vec{v}_\alpha \cdot \vec{v}_\beta}{|\vec{v}_\alpha||\vec{v}_\beta|} = -\frac{1}{3}. \tag{13}$$

By definition this means that vectors $\vec{v}_\alpha$ form a regular tetrahedron. These solutions are not unique and they follow from one another by rotating in the space of the spins and permutations of the indexes α. Therefore, the corresponding determinant for the upper bound is

$$|\Delta| = \frac{1}{12\sqrt{3}}. \tag{14}$$

Having a non-zero determinant for the proposed procedure, ensures its feasibility. One simple choice for vectors $\vec{v}_\alpha$ is

$$\vec{v}_{++} = \frac{1}{4\sqrt{3}}(1,1,1), \qquad \vec{v}_{+-} = \frac{1}{4\sqrt{3}}(-1,1,-1),$$
$$\vec{v}_{-+} = \frac{1}{4\sqrt{3}}(1,-1,-1), \qquad \vec{v}_{--} = \frac{1}{4\sqrt{3}}(-1,-1,1). \tag{15}$$

This yields a simple form for the density matrix of S:

$$\rho_1 = \sqrt{3}\langle \hat{\sigma}_z \rangle, \quad \rho_2 = \sqrt{3}\langle \hat{s}_z \rangle, \quad \rho_3 = \sqrt{3}\langle \hat{s}_z \hat{\sigma}_z \rangle, \tag{16}$$

which gives directly the whole elements of the density matrix, $\hat{\rho}$, in terms of the expectation values and the correlation of the commuting observables $\hat{\sigma}_z$ and $\hat{s}_z$ in the final state.

Next step is to find out the interaction Hamiltonian and the interaction time τ

which give such a description of the tested system, S.

This correspondence can be achieved under the action of the Hamiltonian

$$\hat{H} = \frac{1}{\sqrt{2}}\hat{\sigma}_x(\hat{s}_x\cos\phi + \hat{s}_z\sin\phi) + \frac{1}{2}[(\hat{s}_y - \hat{s}_x)\sin\phi + \hat{s}_z\cos\phi], \qquad (17)$$

where 2ϕ is the angle between $\vec{v}_{++}$ and the z-axis, that is, $\cos\phi = \frac{1}{\sqrt{3}}$. Noting that $\hat{H}^2 = \sin^2\chi$, where χ satisfies $\cos\chi = 1/2\cos\phi$, and taking as duration of the evolution $\tau = \chi/\sin\chi$, we obtain $\hat{U} \equiv exp(-i\hat{H}\tau) = \cos\chi - i\hat{H}$. The simpler form

$$\hat{H} = \frac{1}{\sqrt{2}}\hat{\sigma}_x\hat{s}_x + \frac{1}{2}\left(\hat{s}_y\sin\phi + \hat{s}_z\right) \qquad (18)$$

of $\hat{H}$ can be obtained by a rotation of $\vec{\hat{s}}$ and also achieves an optimal mapping $\hat{\rho} \to P_\alpha$, provided $\hat{s}_z \to \hat{s}_x\sin\phi + \hat{s}_z\cos\phi$ both in the measured projections $\hat{p}_a = 1/2(1\pm\hat{s}_z)$ and in the initial state $\hat{R} = \hat{p}_+$. The first term in (Eq. 18) describes, in the spin language, an Ising coupling, while the second term represents a transverse magnetic field acting on the assistant A.

4. Assistant System as a Coherent State of Light

In this section we discuss the possibility of using light as an assistant to determine the elements of the density matrix of a spin-$\frac{1}{2}$ system. We show that in case of using an electromagnetic field in coherent state, one can determine the state of S from the commutative measurements on S and A.

To describe this physical situation we choose the Jaynes-Cummings model[6,7] a widely accepted model describing the interaction of matter (two-level atom or spin-$\frac{1}{2}$ system) and a single mode of radiation. This model is exactly solvable but still rather non-trivial and it finds direct experimental realization in quantum optics. The Hamiltonian reads:[6]

$$\hat{H} = \hat{H}_A + \hat{H}_S + \hat{H}_{SA} = \hbar\omega\hat{a}^\dagger\hat{a} + \frac{1}{2}\hbar\omega\hat{\sigma}_z + \hbar\gamma(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger) \qquad (19)$$

where $\hat{a}^\dagger$ and $\hat{a}$ are the standard photon creation and annihilation operators of the field (the assistant A), with commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$, $\hat{\sigma}_i$ are the standard Pauli matrices for the spin of the two level system S. The total Hamiltonian (19) is the sum of the Hamiltonian of the field $\hat{H}_A$, the Hamiltonian $\hat{H}_S$ of the two level system S and the interaction Hamiltonian $\hat{H}_{SA}$ which can be written as $\hbar\gamma\hat{V}$, with γ the coupling constant. It can be easily checked that the interaction operator, $\hat{V}$, and the total number of excitations, $\hat{N}$:

$$\hat{V} = \hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger, \quad \hat{N} = \hat{a}^\dagger\hat{a} + \hat{\sigma}_+\hat{\sigma}_-. \qquad (20)$$

are two constants of motion. $\hat{V}$ and $\hat{N}$ also commute with each other since $\hat{V}^2 = \hat{N}$.

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One can then calculate exactly the relevant observables at time t using the Heisenberg equations of motion:

$$
\begin{aligned}
\dot{\hat{a}} &= -i\omega\hat{a} - i\gamma\hat{\sigma}_-, \\
\dot{\hat{\sigma}}_- &= -i\omega\hat{\sigma}_- + i\gamma\hat{\sigma}_z\hat{a}, \\
\dot{\hat{\sigma}}_z &= 2i\gamma(\hat{a}^\dagger\hat{\sigma}_- - \hat{\sigma}_+\hat{a}).
\end{aligned}
\tag{21}
$$

We will briefly outline the result. The exact solution of the above set of equations reads:

$$
\hat{a}(t) = e^{i(\gamma\hat{V}-\omega)t}\left[\left(\cos\gamma\hat{K}t - \frac{i\hat{V}\sin\gamma\hat{K}t}{\hat{K}}\right)\hat{a}(0) - \frac{i\sin\gamma\hat{K}t}{\hat{K}}\hat{\sigma}_-(0)\right],
$$

$$
\hat{\sigma}_-(t) = e^{i(\gamma\hat{V}-\omega)t}\left[\left(\cos\gamma\hat{K}t + \frac{i\hat{V}\sin\gamma\hat{K}t}{\hat{K}}\right)\hat{\sigma}_-(0) - \frac{i\sin\gamma\hat{K}t}{\hat{K}}\hat{a}(0)\right], \quad (22)
$$

where $\hat{K} = \sqrt{\hat{N}+1} = \sqrt{\hat{V}^2+1}$. Note that $\hat{K}$ and $\hat{V}$ commute so that their mutual ordering is irrelevant.

We study the case in which the electromagnetic field is in a coherent state, a condition that coincides with the common experimental situation of a resonant laser mode interacting with a spin-$\frac{1}{2}$ system. Assuming initial factorization between S and A, the density matrix of the total system S+A at time $t = 0$, is:

$$
\hat{\rho}(t=0) = \frac{1}{2}(1 + \langle\hat{\sigma}_i\rangle\hat{\sigma}_i) \otimes \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{e^{-|\alpha|^2}}{\sqrt{k!}\sqrt{m!}}\alpha^k\alpha^{*m}|k\rangle\langle m|
\tag{23}
$$

where $|k\rangle$ denotes the ket for the photon quanta and $|\alpha|^2$ is the average photon number in the coherent state.

We consider as possible triplets of commuting observables the following:

$$
\begin{aligned}
\hat{\sigma}_z &\ , \ \ \hat{a}^\dagger\hat{a} \ , \ \ \hat{\sigma}_z\hat{a}^\dagger\hat{a} \\
\hat{\sigma}_x &\ , \ \ \hat{a}^\dagger\hat{a} \ , \ \ \hat{\sigma}_x\hat{a}^\dagger\hat{a}
\end{aligned}
$$

but others combinations are possible. A direct evaluation shows that only the second choice produces a set of independent equations relating the measurements of the chosen commuting observables at time $t > 0$, after turning on the interaction, and the state of the system S at time $t = 0$. Only in this case then, a reconstruction of the state of the spin-$\frac{1}{2}$ system at time $t = 0$ is possible . We report here the details of the calculation only for this relevant choice of observables. We have to calculate the expectation values of these three observables at time t. In order to perform this calculation, we will make use of the eigenfunctions of the $\hat{V}$ operator, which are:

$$
|\phi_n^\pm\rangle = \frac{|n-1\rangle|+\rangle \pm |n\rangle|-\rangle}{\sqrt{2}} \quad (n \geq 1)\ , \quad |\phi_0^\pm\rangle = |\phi_0\rangle = |0\rangle|-\rangle \quad (n=0), \quad (24)
$$

and the operators $\hat{V}$ and $\hat{K}$, when applied to these functions, evaluate to:

$$
\hat{V}|\phi_n^\pm\rangle = \pm\sqrt{n}\,|\phi_n^\pm\rangle, \qquad \hat{K}|\phi_n^\pm\rangle = \sqrt{n+1}\,|\phi_n^\pm\rangle.
\tag{25}
$$

Let us introduce the following notation:

$$\hat{f}(\hat{V}) = e^{i(\gamma\hat{V}-\omega)t} \Rightarrow f_n^{\pm} = \langle\phi_n^{\pm}|f(\hat{V})|\phi_n^{\pm}\rangle = e^{i(\pm\gamma\sqrt{n}-\omega)t} \rightarrow (f_n^{\pm})^* = \bar{f}_n^{\pm}$$

$$\hat{S}(\hat{K}) = \frac{i\sin\gamma\hat{K}t}{\hat{K}} \Rightarrow S_n = \langle\phi_n^{\pm}|S(\hat{K})|\phi_n^{\pm}\rangle = \frac{i\sin\gamma\sqrt{n+1}t}{\sqrt{n+1}} \rightarrow S_n^* = \bar{S}_n = -S_n$$

$$\hat{g}(\hat{V},\hat{K}) = \cos\gamma\hat{K}t + \hat{V}\hat{S}(\hat{K}) \Rightarrow g_n^{\pm} = \langle\phi_n^{\pm}|\hat{g}(\hat{V},\hat{K})|\phi_n^{\pm}\rangle = \cos\gamma\sqrt{n+1}t \pm \sqrt{n}S_n \rightarrow (g_n^{\pm})^* = \bar{g}_n^{\pm}$$

where, for the sake of compactness, a bar in some cases is used instead of the asterisk
to indicate the complex conjugation operation. In this notation it holds that:

$$\hat{\sigma}_+(t) = (\hat{\sigma}_+ g^{\dagger}(\hat{V},\hat{K}) - \hat{a}^{\dagger}S^{\dagger}(\hat{K}))f^{\dagger}(\hat{V}) \qquad (26)$$

$$\hat{a}^{\dagger}\hat{a}(t) = (\hat{a}^{\dagger}g(\hat{V},\hat{K}) - \hat{\sigma}_+ S^{\dagger}(\hat{K}))(g^{\dagger}(\hat{V},\hat{K})\hat{a} - S(\hat{K})\hat{\sigma}_-)$$

$$\hat{a}^{\dagger}\hat{a}\hat{\sigma}_+(t) = (\hat{a}^{\dagger}g(\hat{V},\hat{K}) - \hat{\sigma}_+ S^{\dagger}(\hat{K}))(g^{\dagger}(\hat{V},\hat{K})\hat{a} - S(\hat{K})\hat{\sigma}_-)(\hat{\sigma}_+ g^{\dagger}(\hat{V},\hat{K}) - \hat{a}^{\dagger}S^{\dagger}(\hat{K}))f^{\dagger}(\hat{V})$$

For a generic observable $\hat{O}$ we define:

$$\langle\hat{O}(t)\rangle = Tr[\hat{\rho}(t=0)\hat{O}(t)] = \sum_{n=0}^{\infty}\sum_{i=\pm}\langle\phi_n^i|\hat{\rho}(t=0)\hat{O}(t)|\phi_n^i\rangle$$

We can then proceed to evaluate the expectation values at a generic time t of the
chosen triplet of commuting observables. As a first step we have:

$$\langle\hat{\sigma}_+(t)\rangle = \sum_{n=0}^{\infty}\frac{e^{-|\alpha|^2}|\alpha|^{2n}}{n!}\left[\left(\frac{\bar{f}_n^+\bar{g}_n^+ - \bar{f}_n^-\bar{g}_n^-}{2\alpha}\sqrt{n} - \bar{S}_n\frac{n-|\alpha|^2}{\alpha}\frac{\bar{f}_n^+ - \bar{f}_n^-}{2}\right)\lambda_1 - \right.$$

$$\bar{S}_n\frac{\bar{f}_n^+ - \bar{f}_n^-}{2}\frac{\alpha^*\sqrt{n}}{\alpha}\lambda_2 + \left(\frac{\bar{f}_n^+\bar{g}_n^+ + \bar{f}_n^-\bar{g}_n^-}{2} - \sqrt{n}\bar{S}_n\frac{\bar{f}_n^+ - \bar{f}_n^-}{2}\right)\lambda_2^* - \left.\bar{S}_n\frac{\bar{f}_n^+ - \bar{f}_n^-}{2}\alpha^*\right]$$

$$\langle\hat{a}^{\dagger}\hat{a}(t)\rangle = \sum_{n=0}^{\infty}\frac{e^{-|\alpha|^2}|\alpha|^{2n}}{n!}\left[\left(|g_n^+|^2(n-|\alpha|^2) - S_n\sqrt{n}\frac{g_n^+ - g_n^-}{2} + |S_n|^2\right)\lambda_1 + \right.$$

$$S_n\frac{g_n^+ + g_n^-}{2}(\alpha\lambda_2^* - \alpha^*\lambda_2) + |\alpha|^2|g_n^+|^2\bigg] \qquad (27)$$

$$\langle\hat{a}^{\dagger}\hat{a}\hat{\sigma}_+(t)\rangle = \sum_{n=0}^{\infty}\frac{e^{-|\alpha|^2}|\alpha|^{2n}}{n!}\left[\frac{n}{\alpha}\left(|g_n^+|^2(\sqrt{n}\frac{\bar{f}_n^+\bar{g}_n^+ - \bar{f}_n^-\bar{g}_n^+}{2} - S_n\frac{\bar{f}_n^+\bar{g}_n^+ - \bar{f}_n^-\bar{g}_n^+}{2}\frac{g_n^+ - g_n^-}{2} + \right.\right.$$

$$S_n\frac{f_n^+ - f_n^-}{2}(n - \frac{|\alpha|^2(n+1)}{n}(1-\frac{1}{\lambda_1})) + S_n\frac{\bar{f}_n^+\bar{g}_n^+ + \bar{f}_n^-\bar{g}_n^-}{2}\frac{g_n^+ + g_n^-}{2}\frac{|\alpha|^2}{n}(1-\frac{1}{\lambda_1})$$

$$+|S_n|^2(\frac{\bar{f}_n^+\bar{g}_n^+ - \bar{f}_n^-\bar{g}_n^-}{2\sqrt{n}} + \frac{\bar{f}_n^+ + \bar{f}_n^-}{2}S_n - \frac{n+1}{\sqrt{n}}\frac{g_n^- + g_n^+}{2}\frac{\bar{f}_n^+ - \bar{f}_n^-}{2} + 2\sqrt{n}\frac{g_n^- - g_n^-}{2}\cdot$$

$$\frac{\bar{f}_n^+ + \bar{f}_n^-}{2} - \frac{|\alpha|^2}{\sqrt{n}}\frac{g_n^- + g_n^+}{2}\frac{\bar{f}_n^+ - \bar{f}_n^-}{2})\bigg)\lambda_1 + \left(n\frac{\bar{f}_n^+\bar{g}_n^+ + \bar{f}_n^-\bar{g}_n^-}{2}|g_n^+|^2 + S_n\frac{\bar{f}_n^+ - \bar{f}_n^-}{2}\cdot\right.$$

$$|g_n^+|^2\sqrt{n} + 2S_n\frac{\bar{f}_n^+\bar{g}_n^+ + \bar{f}_n^-\bar{g}_n^-}{2}\frac{g_n^- - g_n^-}{2} - |S_n|^2(2\frac{g_n^- - g_n^+}{2}\frac{\bar{f}_n^+ - \bar{f}_n^-}{2} + \frac{n+1}{n}\cdot$$

$$\frac{g_n^- + g_n^+}{2}\frac{\bar{f}_n^+ + \bar{f}_n^-}{2} + \frac{\bar{f}_n^+\bar{g}_n^+ + \bar{f}_n^-\bar{g}_n^-}{2n} - \frac{\bar{f}_n^+ - \bar{f}_n^-}{2\sqrt{n}}S_n))\lambda_2^* + n\frac{\alpha^*}{\alpha}\left(|S_n|^2\frac{g_n^- + g_n^+}{2}\cdot\right.$$

$$\frac{\bar{f}_n^- + \bar{f}_n^+}{2} + S_n(|g_n^+|^2\frac{\bar{f}_n^+ - \bar{f}_n^-}{2}\frac{n+1}{\sqrt{n}} - \frac{\bar{f}_n^+\bar{g}_n^+ - \bar{f}_n^-\bar{g}_n^-}{2}\frac{g_n^+ + g_n^-}{2\sqrt{n}}))\lambda_2\bigg]$$

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where we have also defined:

$$\lambda_1 = \frac{1 + \langle \hat{\sigma}_z(0) \rangle}{2}, \quad \lambda_2 = \langle \hat{\sigma}_+(0) \rangle, \quad \lambda_2^* = \langle \hat{\sigma}_-(0) \rangle \tag{28}$$

From Eqs. (27) it is then easy to derive a system of three equations relating the three expectation values:

$$\langle \hat{\sigma}_x(t) \rangle = 2\Re[\langle \hat{\sigma}_+(t) \rangle], \quad \langle \hat{a}^\dagger \hat{a}(t) \rangle, \quad \langle \hat{a}^\dagger \hat{a} \hat{\sigma}_x(t) \rangle = 2\Re\left[\langle \hat{a}^\dagger \hat{a} \hat{\sigma}_+ t \rangle\right] \tag{29}$$

evaluated at a generic time t, to the variables $\langle \hat{\sigma}_x(0) \rangle, \langle \hat{\sigma}_y(0) \rangle, \langle \hat{\sigma}_z(0) \rangle$, i.e. to the density matrix of the spin-$\frac{1}{2}$ system at time $t = 0$. In order to assess if this system of equations has solutions, we have to evaluate the determinant of the matrix $\hat{M}$ made up by the coefficients of $\langle \hat{\sigma}_x(0) \rangle, \langle \hat{\sigma}_y(0) \rangle, \langle \hat{\sigma}_z(0) \rangle$ appearing in the system of equations determined by the evaluation of (29). In order to achieve this goal, it is convenient to define $\hat{M}_{ij} = \sum_{n=0}^{\infty} \frac{e^{-|\alpha|^2} |\alpha|^{2n}}{n!} M_{ij}(n)$ and then determine the coefficients $M_{ij}(n)$. Name $A_{ij}(n)$ the same type of coefficients but relative to the matrix $\hat{A}$ made up by the coefficients multiplying the variables $\lambda_1, \lambda_2, \lambda_2^*$ in the system of equations (27). By replacing the definitions of $f_n^\pm, g_n^\pm, S_n$ and then simplifying, one gets:

$$A_{11}(n) = -i\frac{e^{i\omega t}}{\alpha}\left(\sqrt{n}\cos\gamma\sqrt{n+1}t\sin\gamma\sqrt{n}t + |\alpha|^2\frac{\cos\gamma\sqrt{n}t\sin\gamma\sqrt{n+1}t}{\sqrt{n+1}}\right) \tag{30}$$

$$A_{12}(n) = \frac{ie^{i\omega t}\sqrt{n}}{\alpha\sqrt{n+1}}\sin\gamma\sqrt{n+1}t\sin\gamma\sqrt{n}t$$

$$A_{13}(n) = e^{i\omega t}\cos\gamma\sqrt{n+1}t\cos\gamma\sqrt{n}t$$

$$A_{21}(n) = \frac{(1+2n)(1+n-|\alpha|^2) - (1+n+|\alpha|^2)\cos 2\gamma\sqrt{n+1}t}{2(n+1)}$$

$$A_{22}(n) = \frac{i\alpha\cos\gamma\sqrt{n+1}t\sin\gamma\sqrt{n}t}{\sqrt{n+1}}, \quad A_{23}(n) = A_{22}^*(n)$$

$$A_{31}(n) = nA_{11}(n), \quad A_{32}(n) = nA_{13}(n), \quad A_{33}(n) = nA_{12}(n)$$

It is easy to check that these coefficients are related to the coefficients $M_{ij}(n)$ in the following way :

$$M_{i1}(n) = \Re[A_{i1}(n)], \quad M_{i2}(n) = \Re[A_{i2}(n) + A_{i3}^*(n)], \quad M_{i3}(n) = \Im[A_{i2}(n) + A_{i3}^*(n)] \tag{31}$$

If we now calculate the determinant $\Delta(t)$ of the matrix $\hat{M}$, we obtain:

$$\Delta(t) = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{e^{-3|\alpha|^2} |\alpha|^{2(l+n+m)}}{l!m!n!} \varepsilon_{ijk} M_{1i}(l) M_{2j}(m) M_{3k}(n) \tag{32}$$

We evaluate this determinant numerically. Convergence within the seventh significant figure is reached by keeping in each of the three sums the first 30 terms when $|\alpha|^2 \le 9$. For larger values of $|\alpha|^2$ convergence turns out to be much slower. In Fig. 1 the temporal evolution of the determinant is shown. We see that as $|\alpha|^2$ increases, i.e. for larger average number of photons, the determinant has fluctuations of larger

amplitude, so that a time with a large enough determinant can be chosen to solve for the initial state of the spin-$\frac{1}{2}$ system. When the matrix elements are determined with some experimental uncertainty such a choice is a sensible one that allows a more accurate determination of the initial state of the spin-$\frac{1}{2}$ system and avoids cases of ill-conditioned matrix inversion.
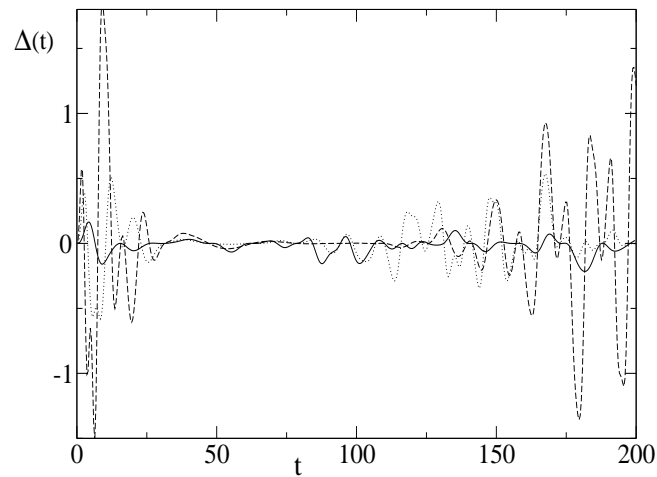

Fig. 1.   Evolution of the determinant for $0 < t < 200$, with the following choice of parameters of the model: $\gamma = 0.1, \omega = 0.1$. The continuous line refers to the case $|\alpha|^2 = 1$ the dotted line to $|\alpha|^2 = 4$, the dashed line to $|\alpha|^2 = 9$, with α real and positive.

5. Conclusions

We have illustrated a procedure which allows to reconstruct the state of a spin-$\frac{1}{2}$ system with a simultaneous measurement of the expectation values of three commuting observables, by coupling the system to an assistant. We have also illustrated how the procedure works in the simple case of a spin-$\frac{1}{2}$ system coupled to a coherent laser field. We have shown that in this case, after a proper choice of the commuting observables, it is always possible to reconstruct the initial state of the spin-$\frac{1}{2}$ system. A radiation source with a large average number of photons allows to implement the procedure in an experimentally reliable condition, i.e. with a large absolute value of the determinant of the matrix connecting the expectation values of the commuting observables at time t to the initial state of the spin-$\frac{1}{2}$ system.

Acknowledgments

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